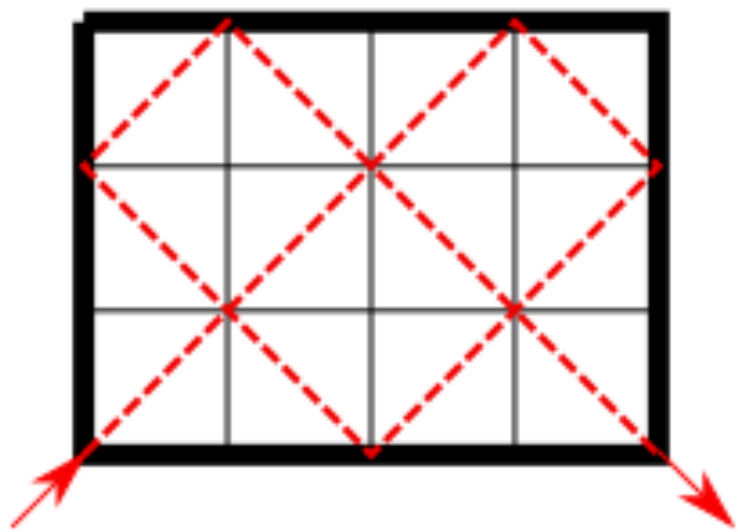
Puzzle of the Week
Bouncing Billiard Ball

THE CHALLENGE: A billiard ball shot at a 45 degree angle from the lower left corner of a 3 by 4 billiard table will bounce 5 times before hitting the lower right corner of the table. If you start at the lower left corner of a 4 by 7 billiard table, how many times will the ball bounce before hitting one of the corners?



EXPLORATION: For other sizes of tables, can you predict which corner will be hit first and how many bounces will be made? Can you use some ideas from geometry to help you simplify the problem?

Puzzle of the Week

Bouncing Billiard Ball – Notes

THE CHALLENGE & EXPLORATION: This can be played with and thoroughly enjoyed by using a piece of graph paper and seeing what happens. It can also be approached more systematically and rigorously.

The first thing to notice is that the size of the table is not as important as its shape. Any rectangle that is similar to another (has the same ratio of side lengths) will give the same result. So a 3 by 4 table will be the same as a 6 by 8 table and a 9 by 12 table.

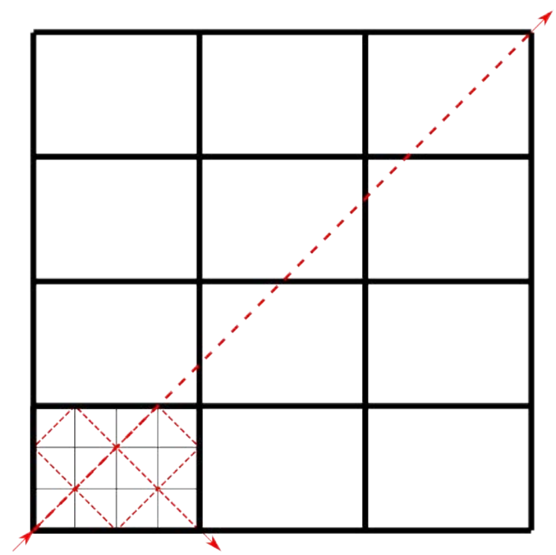
A useful exercise at this point is to collect the results of what happens for 1 by n and 2 by n tables.

- 1 by n: The ball will take n - 1 bounces and will end up in the bottom right corner for n even, and top right corner if n is odd.
- 2 by n: If n = 2k, then 2 by 2k behave like a 1 by k table (because they are similar). A 2 by (2k + 1) there are n bounces and the ball ends up in the upper left corner.

After some more experimentation a general result appears!

<u>Result:</u> If k and n have no factors in common, then a k by n table will have k + n - 2 bounces. The ball will end in the upper left corner if k is even. If k is odd, the ball will end in the upper right corner if n is odd and in the lower right hand corner if n is even. If k and n do have common factors, then divide out those factors and apply this result to the reduced numbers.

One neat technique to see this more easily is to “unfold” the bounces. Make a grid of tables, as pictured, and let the ball go in a straight line through all the table tops!

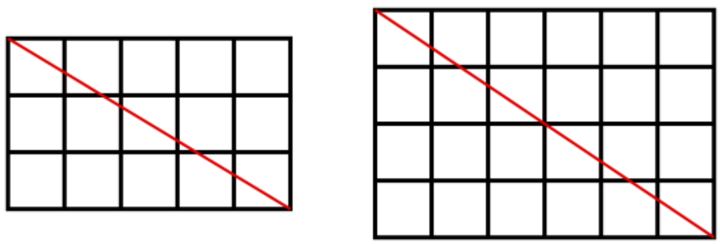


Puzzle of the Week
Crossing Lines

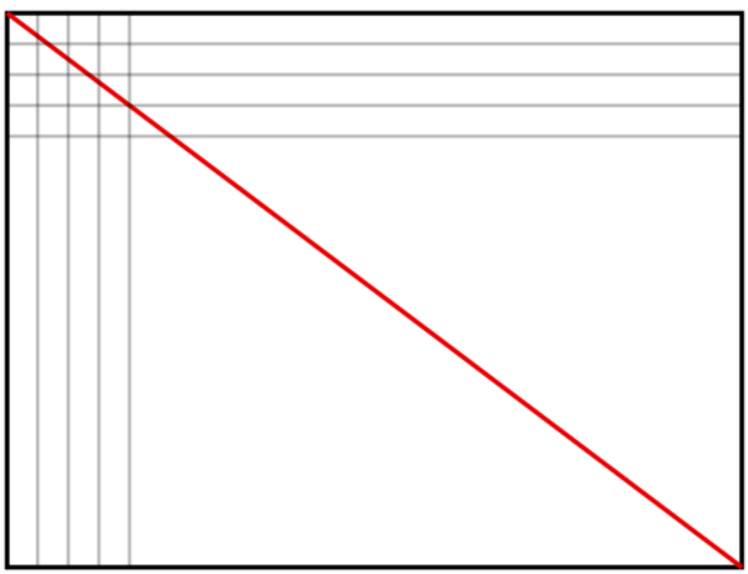
These two rectangles have been cut in half using a red line.

In the 3 by 5 rectangle: the red line crosses 6 sides of small squares. Check to see you agree!

In the 4 by 6 rectangle: the red line also crosses 6 sides of small squares. (We're not counting the place where the line goes through the corner of a square.)



THE CHALLENGE: Here's a 36 by 42 rectangle that's been cut in half by a red line. How many sides of small squares will the red line cross?

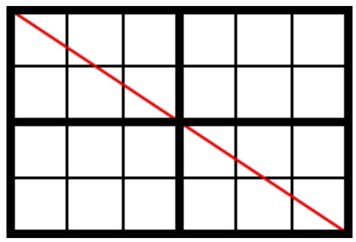


EXPLORATION: What happens for rectangles of other sizes? What patterns do you see?

Puzzle of the Week

Crossing Lines – Notes

THE CHALLENGE & EXPLORATION: A diagonal will go through a corner of a box exactly when the change in height and width evenly divides the height and width of the rectangle..



Consider the 4 by 6 box in the introduction. Because 2 is a common factor of 4 and 6, we can break the larger 4 by 6 box into a 2 by 2 collection of 4/2 by 6/2 boxes. The diagonal of the 4 by 6 box will be the diagonal in 2 of the 2 by 3 boxes and go through the corner between them.

In the other introductory example of the 3 by 5 rectangle, the greatest common divisor is 1, so the diagonal does not go through any corners.

Suppose the rectangle is n by m, and that the greatest common divisor of n and m is c.

> **c = 1:** In this case, the diagonal will cross all n - 1 internal horizontal lines and all m - 1 internal vertical lines. So, it will cross a total of n + m - 2 box sides.
>
> **c > 1:** Break the diagonal into c pieces. Each piece of the diagonal will be a diagonal for a n/c by m/c rectangle, where the greatest common factor of n/c and m/c is 1. So, each piece of the diagonal will cross n/c + m/c - 2 box sides. All the pieces together will cross c x (n/c + m/c - 2) = n + m - 2c box sides.

The general answer is n + m - 2c. Let’s apply this to the two introductory examples and the challenge puzzle.

For a 3 by 5 rectangle, c will be 1 and the answer is 3 + 5 - (2 x 1) = 8 – 2 = 6.

For a 4 by 6 rectangle, c will be 2 and the answer is 4 + 6 - (2 x 2) = 10 - 4 = 6.

For the challenge 36 by 42 rectangle, c will be 6 and the answer is 36 + 42 - (2 x 6) = 78 - 12 = 66.

Puzzle of the Week

Egyptian Fractions – 1

Around 4000 years ago, the ancient Egyptians developed a special way of writing fractions. ***Unit Fractions***, which are fractions with 1 in the numerator such as $\frac{1}{3}$ and $\frac{1}{8}$, were important to them, and are also known as ***Egyptian Fractions***. The Egyptians wrote *any* fractional quantity as an ***Egyptian Fraction Sum***, which is a sum of Egyptian Fractions with no duplicates. For example, for $\frac{7}{8}$ they wrote the Egyptian Fraction Sum $\frac{7}{8} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8}$.

THE CHALLENGE: Write 1 as an Egyptian Fraction Sum using as few fractions as possible.

$$1 = 1/A + 1/B + \ldots$$

EXPLORATION: Convince yourself that you can't use fewer fractions to get 1. Write $\frac{2}{3}$, 3/2, 2/7, 3/7, and 4/7 as an Egyptian Fraction Sum. Play with other numbers and look for patterns for ways to break a fraction into Egyptian Fractions that will help you work with them.

Puzzle of the Week

Egyptian Fractions – 1 – Notes

THE CHALLENGE: The answer for this is simple enough: 1 = ½ + ⅓ + ⅙.

EXPLORATION: The only way to use two fractions would be as 1 = ½ + ½, and that is not allowed.

A ***Greedy Algorithm*** can be used for creating Egyptian Fraction Sums. At each step, this algorithm picks the unit fraction with the smallest possible denominator (the largest possible value for the fraction). Using that approach, ⅔ = ½ + ⅙; 2/7 = ¼ + 1/28; 3/7 = ⅓ + 1/11 + 1/231; and 4/7 = ½ + 1/14. The Greedy Algorithm can lead to some needlessly large denominators, but it always works. For example, 3/7 = ⅓ + 1/11 + 1/231, but it can be written more simply as 3/7 = ¼ + 1/7 + 1/28.

One way to do 3/2 would be to start with 1 = ½ + ⅓ + ⅙, and combine that with 1⁄2 = ¼ + ⅕ + 1/20. Doing this, we end up with 3/2 = 1 + ½ = (½ + ⅓ + ⅙) + (¼ + ⅕ + 1/20). Your students may find other interesting ways to do this.

If you go to Wikipedia, you will find many interesting formulas for Egyptian Fraction Sums. One useful formula for 1/n is 1/(n + 1) + 1/[n(n+1)]. For example, ¼ = ⅕ + 1/20.

Puzzle of the Week

Egyptian Fractions – 2

Around 4000 years ago, the ancient Egyptians developed a special way of writing fractions. ***Unit Fractions***, which are fractions with 1 in the numerator such as $\frac{1}{3}$ and $\frac{1}{8}$, were important to them, and are also known as ***Egyptian Fractions***. The Egyptians wrote *any* fractional quantity as an ***Egyptian Fraction Sum***, which is a sum of Egyptian Fractions with no duplicates. For example, for $\frac{7}{8}$ they wrote the Egyptian Fraction Sum $\frac{7}{8} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8}$.

THE CHALLENGE: Write 39/50 as an Egyptian Fraction Sum using as few fractions as possible.

$$39/50 = 1/A + 1/B + \ldots$$

EXPLORATION: Is your answer the absolute best? If yes, what are your reasons?

Puzzle of the Week

Egyptian Fractions – 2 – Notes

THE CHALLENGE: You can chip away at this by subtracting the largest Egyptian Fraction you can and see what is left over. This always works with any fraction, but it doesn't always produce the fewest number of Egyptian Fractions.

First, 39/50 > ½, so 39/50 = ½ + 14/50 = ½ + 7/25.

7/25 > ¼, so subtract ¼ off next. 39/50 = ½ + ¼ + 3/100.

Finally, 3/100 = 2/100 + 1/100 = 1/50 + 1/100.

Putting it all together, 39/50 = ½ + ¼ + 1/50 + 1/100.

EXPLORATION: I am not aware of a shortcut for analyzing whether fewer fractions are possible.

Consider if it were possible to use three fractions. The fraction with the largest value must be at least ⅓. Otherwise, the largest value for the sum of the three fractions would be ¼ + ⅕ + ⅙, and that is not big enough.

- The largest fraction is ½: The remaining value is 39/50 - ½ = 7/25. The larger of the two remaining fractions must be at least 1/7. That leads to four possibilities to look at: ½ + ¼, ½ + ⅕, ½ +⅙, and ½ + 1/7.
- The largest fraction is ⅓: The remaining value is 39/50 - ⅓ = 67/150. The larger of the two remaining fractions must be at least ⅕. That leads to two possibilities to look at: ⅓ + ¼ and ⅓ + ⅕.

It's not pretty. However, there are not that many combinations to check to see that three fractions will not work.

Puzzle of the Week
Egyptian Fractions – 3

Around 4000 years ago, the ancient Egyptians developed a special way of writing fractions. ***Unit Fractions***, which are fractions with 1 in the numerator such as $\frac{1}{3}$ and $\frac{1}{8}$, were important to them, and are also known as ***Egyptian Fractions***. The Egyptians wrote *any* fractional quantity as an ***Egyptian Fraction Sum***, which is a sum of Egyptian Fractions with no duplicates. For example, for $\frac{7}{8}$ they wrote the Egyptian Fraction Sum $\frac{7}{8} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8}$.

THE CHALLENGE: Write each of the fractions 1/20, 2/20. 3/20, 4/20, 5/20, 6/20, 7/20, 8/20, 9/20, and 10/20 as an Egyptian Fraction or an Egyptian Fraction Sum using only two fractions. If the fraction is already an Egyptian Fraction, such as 1/20, you can use it as is.

$$X/20 = 1/A$$

or

$$X/20 = 1/A + 1/B$$

EXPLORATION: Play around with other groups of fractions with the same denominator (e.g. 6, 7, 8, 12, 17, 21, 50, etc.). For which denominators do you need to create Egyptian Fraction Sums with more than two fractions?

Puzzle of the Week

Egyptian Fractions – 3 – Notes

THE CHALLENGE: Here is the list of results:

- 1/20 = 1/20
- 2/20 = 1/10
- 3/20 = 2/20 + 1/20 = 1/10 + 1/20
- 4/20 = ⅕
- 5/20 = ¼
- 6/20 = 3/10 = 2/10 + 1/10 = ⅕ + 1/10
- 7/20 = 5/20 + 2/20 = ¼ + 1/10
- 8/20 = ⅖ = ⅓ + 1/15
- 9/20 = 5/20 + 4/20 = ¼ + ⅕
- 10/20 = ½

EXPLORATION: Here are a few more families of fractions. The ones whose denominators create fractions that are more likely to reduce are easier to do. In particular, denominators that are primes tend to be messy.

- ⅙ = ⅙
- 2/6 = ⅓
- 3/6 = ½
- 4/6 = ⅔ = ½ + ⅙
- ⅚ = ½ + ⅓

- 1/7 = 1/7
- 2/7 = ¼ + 1/28
- 3/7 = ¼ + 5/28 = ¼ + 1/7 + 1/28
- 4/7 = ½ + 1/14
- 5/7 = ½ + 3/14 = ½ + 1/7 + 1/14
- 6/7 = ½ + 5/14 = ½ + ⅓ + 1/42

- ⅛ = ⅛
- 2/8 = ¼
- ⅜ = ¼ + ⅛
- 4/8 = ½
- ⅝ = ½ + ⅛
- 6/8 = ¾ = ½ + ¼
- ⅞ = ½ + ¼ + ⅛

Puzzle of the Week

Egyptian Fractions – 4

Around 4000 years ago, the ancient Egyptians developed a system where all their fractions were of the form 1/n. These are called ***Egyptian Fractions***. Some Egyptian Fractions can be written as the difference of two Egyptian Fractions. $\frac{1}{3} = \frac{1}{2} - \frac{1}{6}$ is an example of doing that.

THE CHALLENGE: Which Egyptian Fractions can be written as a difference of two Egyptian Fractions in only one way? Which ones can be written in exactly two ways? Are there some that have more than two ways?

$$1/N = 1/A - 1/B$$

Puzzle of the Week

Egyptian Fractions – 4 – Notes

THE CHALLENGE: One thing to note is that for any n, there are a limited number of possible solutions. If 1/N = 1/A - 1/B, then 1/A - 1/N = 1/B > 0. 1/A - 1/N > 0 means that N > A. So, 1 < A < N, which means there are at most N - 2 solutions, and they can be found, somewhat tediously, by checking all the values of A in that range.

For denominators that are even numbers, say 2d, there is always 1/2d = 1/d - 1/2d.

Another way to produce differences is to start with an addition formula. The simplest one we have is that 1/n = 1/(n+1) + 1/[n(n+1)]. This becomes 1/(n+1) = 1/n - 1/[n(n+1)]. A more general version of this is 1/ab = 1/[a(a+b)] + 1/[b(a+b)] which becomes 1/[a(a+b)] = 1/ab - 1/[b(a+b)] where ab > 1.

So, whenever the denominator can be factored as a(a+b), with a and b positive and ab > 1, we will have a difference. In particular, because every number n > 2 can be written as n = 1 x (1 + (n-1)), then every Egyptian Fraction can be written as a difference in at least one way. Here are a few examples:

- ⅓: a = 1, b = 2 gives ½ - ⅙
- ¼: a = 1, b = 3 gives ⅓ - 1/12
- ⅕: a = 1, b = 4 gives ¼ - 1/20
- ⅙: a = 1, b = 5 gives ⅕ - 1/30; a = 2, b = 1 gives ½ - ⅓
- 1/7: a = 1, b = 6 gives ⅙ - 1/42
- ⅛: a = 1, b = 7 gives 1/7 - 1/56; a = 2, b = 2 gives ¼ - 1/8

This is not everything. For example, we have missed ⅙ = ¼ - 1/12.

Diving into some algebra, for a > 1 and b > a >1, consider 1/n = 1/a - 1/b = (b - a) / ab. Clearing the denominators produces ab = n(b-a), and solving for n gives, n = ab/(b-a). Whenever we can find a and b that makes this work, we will be able to produce the difference.

Let's look at the complete solutions for a few examples from this point of view and see if we can spot a pattern.

- 3 = 2 x 6 / (6 - 2)
- 4 = 2 x 4 / (4 -2) = 3 x 12 / (12 - 3)
- 5 = 4 x 20 / (20 - 4)
- 6 = 2 x 3 / (3 - 2) = 3 x 6 / (6 - 3) = 4 x 12 / (12 - 4) = 5 x 30 / (30 - 5)
- 7 = 6 x 42 / (42 - 6)
- 8 = 4 x 8 / (8 - 4) = 6 x 24 / (24 - 6) = 7 x 56 / (56 - 7)

Looking at these examples, it seems likely that there is only one solution when n is a prime. Let's see why that is. If n is a prime and a < n, then n = a x b / (b - a) forces b to be a multiple of n, say b = n x c. Then n = a x nc / (nc - a) says that ac / (nc - a) = 1, which means ac = nc - a. Rewriting this we get nc = a(c + 1). Because n is a prime, we have c + 1 = n and a = c = n - 1. So a = n - 1 and b = n(n-1) is the only solution!

Puzzle of the Week

Equal Products

THE CHALLENGE Find seven different single digits that make these three products the same.

$$A \times B \times C = C \times D \times E = E \times F \times G$$

1 2 3 4 5 6 7 8 9

EXPLORATION: After you solve this, think about what makes this an interesting puzzle. How might you change the puzzle and still keep it interesting? What happens if you use addition instead of multiplication?

Puzzle of the Week

Equal Products – Notes

THE CHALLENGE: First notice that 5 cannot be one of the digits. If it were, it could be in at most two of the groups of three - this would mean one or two of the groups would have a factor of 5, but the remaining one(s) would not. Similarly, 7 cannot be one of the digits.

That leaves the seven digits, 1, 2, 3, 4, 6, 8, and 9, all of which must be used.

Using prime factorizations makes it a lot easier to understand what happens in the remainder of the analysis. We will need to balance the number of factors of 2 and the number of factors of 3 among the three products.

The product of these seven digits is $2^7 x\ 3^4$. What must the common product be? It must be at least $2^3 x\ 3^2$ as there will be an 8 and a 9 somewhere in the solution. If we multiply out the three groups, that creates a total product of $2^9 x\ 3^6$. Because C and E are the only repeated letters in multiplying together the three products, that means $CxE = 2^2 3^2$. The only way to do this is to have C and E be 4 and 9.

A quick check reveals that increasing the power of 2 or 3 will not work out.

Once we set C = 9 and E = 4, the rest follows quickly.

The answer is: 1 8 9 2 4 3 6.

EXPLORATION: What makes this puzzle interesting is the role prime numbers play in it. Also, the selection of numbers is two more than the number of digits needed, and we are able to rule out exactly two of the possibilities.

If we were to use addition instead of multiplication, there would be way too many solutions. A good puzzle incorporates constraints that narrow it down to having just a few solutions.

Puzzle of the Week

Extreme Products – 2

THE CHALLENGE: 1) Using the digits from 1 to 9, each at most once, make two 3-digit numbers whose product is as large as possible. 2) Also, using the digits from 1 to 9, each at most once, make two 3-digit numbers whose product is as small as possible.

□ □ □ × □ □ □

1 2 3 4 5 6 7 8 9

EXPLORATION: Can you apply what you learned for multiplying two numbers to do this with multiplying three 3-digit numbers? Can you think of other interesting variations?

Puzzle of the Week

Extreme Products – 2 – Notes

THE CHALLENGE: Each higher order place has a much larger effect than the lower order places.

To make the product large we will want 8 and 9 for the hundreds place, 6 and 7 for the tens place, and 4 and 5 in the ones place. Some experimentation reveals that 964 x 875 gives the maximum value of 843,500.

Similar logic for making the product small produces the answer 135 x 246 = 33,210.

EXPLORATION: For three numbers, the analysis is similar, though there are a lot of possibilities.

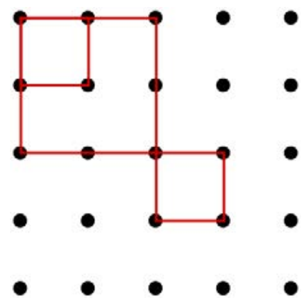
For making the product large, the trend has been to make the numbers associated with 9 smaller and the ones associated with 7 larger. That produces 941 x 852 x 763 = 611,721,516, which is the answer.

Similarly, to make the product small, make the numbers associated with 1 larger and the ones associated with 3 smaller. That produces 147 x 258 x 369 = 13,994,694, which is the answer.

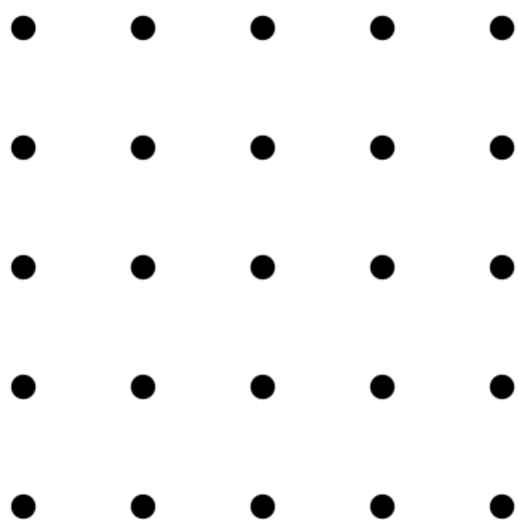
Puzzle of the Week

Finding Squares – 3

Drawn in red in this grid are two 1 by 1 squares and one 2 by 2 square with horizontal and vertical sides.



THE CHALLENGE: FInd all the different sizes of squares you can in this grid. Note that there are some squares that do **not** have horizontal and vertical sides. For each size square you find, what is its area? Once you know its area, what is its side length?



EXPLORATION: Can you discover different methods for finding these areas? Can you find a formula for the side length using just the description of one side?

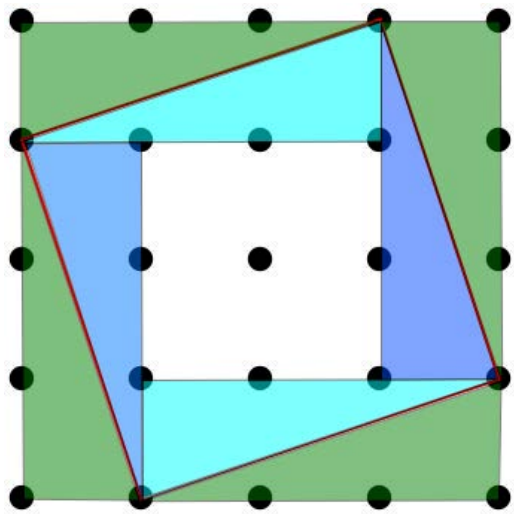
Puzzle of the Week

Finding Squares – 3 – Notes

THE CHALLENGE & EXPLORATION: Finding the area of the squares with horizontal and vertical sides is easy enough. There are four of these of sizes 1 by 1, 2 by 2, 3 by 3, and 4 by 4, with areas 1, 4, 9, and 16.

The other squares are a bit trickier. The insight is to cut the region into pieces whose area we know. Alternatively, we can surround the square by a bigger square and look at the pieces inside it.

Let's look at a square formed by a side that goes over three and up one.



We can find the area by putting together the four interior blue right triangles together with the white square. That produces an area of 4 x (½ x 3 x 1) + 2 x 2 = 6 + 4 = 10. The side length will be the square root of 10.

We can also find the area by taking the big outer square and subtracting off the four green right triangles. That produces an area of 4 x 4 - (4 x (½ x 3 x 1)) = 16 - 6 = 10, just as before.

A similar process can be used for any of the diagonal squares.

After doing several of these, the interested student may notice a pattern in this. Your students' algebra may not be up to this, but a pattern with the numbers, if laid out in a table, can still be seen in lieu of using letters. Let "a" is the amount the side goes to the right, and "b" is the amount the side goes up. Finding the area from the inside amounts to adding 4 x (½ x a x b) + (a - b) x (a - b), which is $a^2 + b^2$. Finding the area from the outside yields (a + b) x (a + b) - 4 x (½ x a x b), which simplifies to the same result.

What we have done is found two proofs for the Pythagorean Theorem!

Puzzle of the Week

Fractions – 1

THE CHALLENGE: Use the numbers 1 to 9 at most once each to fill in the boxes. Can you find more than one solution?

$$\frac{\square}{\square} = \frac{\square}{\square} = \frac{\square}{\square\square}$$

1 2 3 4 5 6 7 8 9

Puzzle of the Week

Fractions – 1 – Notes

THE CHALLENGE: To help limit the search, notice that improper fractions can't be used - the rightmost fraction has a value less than 1. Note that 5's can't possibly work anywhere as they would require a 0 or a second 5.

Look for solutions by running through pairs of denominators that would create the two equal single-digit fractions. There are X's next to the ones that don't work out.

- 4 & 8: 1/4 = 2/8 X; 2/4 = 4/8 X; 3/4 = 6/8 = 9/12
- 2 & 8: 1/2 = 4/8 X
- 6 & 8: 3/6 = 4/8 X
- 3 & 6: 1/3 = 2/6 X; 2/3 = 4/6 X
- 4 & 6: 2/4 = 3/6 = 9/18

To summarize, there are a total of two solutions:

3/4 = 6/8 = 9/12

2/4 = 3/6 = 9/18

Puzzle of the Week

Fractions – 2

THE CHALLENGE: Use the numbers 1 to 9 at most once each to fill in the boxes. Make $\frac{\square}{\square} + \frac{\square}{\square}$ as small as possible. Also, use the numbers 1 to 9 at most once each to make $\frac{\square}{\square} + \frac{\square}{\square}$ as large as possible.

$$\frac{\square}{\square} + \frac{\square}{\square}$$

1 2 3 4 5 6 7 8 9

EXPLORATION: What happens if the fractions must be proper fractions?

Puzzle of the Week
Fractions – 2 – Notes

THE CHALLENGE: To make the sum small, use large denominators to make each fraction small. The denominators should be 8 and 9, and the numerators will be 1 and 2.

Which numerator should go with which denominator? 1/8 + 2/9 or 2/8 + 1/9? This is easy to see if you think of this as 1/8 + 1/9 plus either 1/8 or 1/9. To increase it by as little as possible, choose 1/9.

So, to make it as small as possible, use 1/8 + 2/9.

To make it as large as possible, reverse the roles. Put 8 and 9 on top and 1 and 2 in the bottom. Once again, if you consider what will make the bigger increase to 8/1 + 8/2, you see it will be turning this into 9/1 + 8/2 = 13.

EXPLORATION: Insisting that the fractions are proper does not change the answer for the smallest value.

To get the largest possible value with proper fractions, use large denominators and numerators that are slightly less than the denominator. That means considering 8/9, 7/8, 6/7, 7/9, and 6/8. After a small bit of playing around, 8/9 + 6/7 is the clear winner.

Puzzle of the Week

Fractions – 3

THE CHALLENGE: Use the numbers 1 to 9 at most once each to fill in the boxes. FInd two mixed numbers, whose fractional parts are proper fractions, that have as small a difference as possible.

$$\square \frac{\square}{\square} - \square \frac{\square}{\square}$$

1 2 3 4 5 6 7 8 9

EXPLORATION: How does your answer change if you allow the fractional parts to be improper?

Puzzle of the Week

Fractions – 3 – Notes

THE CHALLENGE & EXPLORATION:

If you allow the fractional parts to be improper, then the difference can be 0. For example, 8 1/2 – 7 6/4.

If they must be proper fractions, then make the first fractional part as small as possible and the second as large as possible.

There are only a few possibilities that have any reasonable chance.

3 1/9 – 2 7/8. For this, the difference is 1/9 + 1/8.

3 1/8 – 2 7/9. For this, the difference is 1/8 + 2/9

3 1/7 – 2 8/9. For this, the difference is 1/7 + 1/9

The first one is the best. Clearly, the 3 and 2 could be replaced by other consecutive numbers without changing the result.

Puzzle of the Week

Fractions – 4

THE CHALLENGE: Use the numbers 1 to 9 at most once each to fill the boxes in each of these puzzles. Make these two-digit fractions as close to the target numbers from 1 to 5 as possible, without equalling the number.

For example, 85/ 17 = 5 is not allowed as a contender, but 83 / 17 is okay and is somewhat close to 5.

$$\frac{\square\square}{\square\square} \sim 1 \qquad \frac{\square\square}{\square\square} \sim 2$$

$$\frac{\square\square}{\square\square} \sim 3 \qquad \frac{\square\square}{\square\square} \sim 4$$

$$\frac{\square\square}{\square\square} \sim 5$$

1 2 3 4 5 6 7 8 9

EXPLORATION: Look at what happens for targets from 6 to 10. What changes in your strategy for targets above some value?

Puzzle of the Week

Fractions – 4 – Notes

THE CHALLENGE & EXPLORATION: The target number of 1 is a different case from the other targets.

To get close to 1, we need the numerator and denominator to be as close as possible. Because they have different tens digits, and we cannot use 0's, the closest we can get is two apart - for example, 29 and 31 yields 29/31 or 31/29. The distance away from 1 will be 2 over the denominator. Keeping with the example, the differences with 1 are 2/31 and 2/29.

With 2 as a constant numerator, we want to make the denominator as large as possible to get close to 1. 91/89 does not work as it uses 9 twice. 81/79 is therefore the best answer.

It is simpler for the target numbers larger than 1. Take 6 as an example target. 96/16 = 6 exactly. To get close to that we have a choice of 97/16 or 95/16, which are equally close, both being 1/16 away.

The answers are:

- 1 81/79
- 2 95/48 or 97/48
- 3 95/32 or 97/32
- 4 95/24 or 97/24
- 5 91/18
- 6 95/16 or 97/16
- 7 97/14
- 8 95/12 or 97/12

Above a target of 8, the best we can do is 98/12 no matter what the target number is. 98 cannot be made any larger, and 12 cannot be made any smaller.

Puzzle of the Week
Fractions – 5

Example: 2/5 is less than 1/2, 19/38 equals 1/2, and 4/7 is larger than 1/2. The difference between 4/7 and 2/5 is (4/7 - 1/2) + (1/2 - 2/5) = 1/14 + 1/10, which is not particularly small. The fractions 2/5, 19/38, and 4/7 use the digits 1 to 9 at most once each.

THE CHALLENGE: Use the numbers 1 to 9 at most once each to fill in these boxes. Make the difference between the largest and smallest fraction as small as possible.

$$\frac{\square}{\square} < \frac{1}{2}$$

$$\frac{\square\square}{\square\square} = \frac{1}{2}$$

$$\frac{\square}{\square} > \frac{1}{2}$$

1 2 3 4 5 6 7 8 9

Puzzle of the Week

Fractions – 5 – Notes

THE CHALLENGE: After some initial exploring, it is clear that 1/14 + 1/18 is the best possible difference. That difference comes from using denominators of 7 and 9, and there are no better ones. The only ways to get that difference are: 5/9 - 3/7 or 4/7 - 4/9, and the second one is not allowed because four is repeated.

If 3, 5, 7, and 9 are used on the outside fractions, that only leaves 1, 2, 4, 6, and 8 to be used for the middle fraction. There are only two ways to get 1/2 using those numbers without repeating a digit: 14/28 and 41/82.

This leaves two entries tied for first place:

3/7 < 14/28 = 1/2 < 5/9

3/7 < 41/82 = 1/2 < 5/9

Puzzle of the Week

Fractions – 6

THE CHALLENGE: Use the numbers 1 to 9 at most once each to fill in these boxes. How many solutions can you find?

$$\frac{\square}{\square} + \frac{\square}{\square} = \square$$

1 2 3 4 5 6 7 8 9

EXPLORATION: Are there any single-digit values for the right side of the equation that are impossible?

Puzzle of the Week
Fractions – 6 – Notes

THE CHALLENGE:

Some trial an error with compatible denominators produces this list:

- 2/4 + 3/6 = 1
- 3/4 + 2/8 = 1
- 3/6 + 4/8 = 1
- 4/6 + 3/9 = 1
- 4/3 + 6/9 = 2
- 5/4 + 6/8 = 2
- 9/6 + 4/8 = 2
- 5/2 + 4/8 = 3
- 9/4 + 6/8 = 3
- 2/1 + 6/3 = 4
- 5/2 + 9/6 = 4
- 7/2 + 3/6 = 4
- 2/1 + 9/3 = 5
- 3/1 + 4/2 = 5
- 3/1 + 8/4 = 5
- 4/2 + 9/3 = 5
- 6/2 + 8/4 = 5
- 7/2 + 6/4 = 5
- 7/2 + 9/6 = 5
- 9/2 + 3/6 = 5
- 9/2 + 4/8 = 5
- 9/3 + 8/4 = 5
- 3/1 + 8/2 = 7
- 4/1 + 6/2 = 7
- 4/1 + 9/3 = 7
- 5/1 + 6/3 = 7
- 5/1 + 4/2 = 7
- 5/1 + 8/4 = 7
- 9/3 + 8/2 = 7
- 5/1 + 6/2 = 8
- 5/1 + 9/3 = 8
- 6/1 + 4/2 = 8
- 5/1 + 8/2 = 9
- 7/1 + 4/2 = 9
- 7/1 + 6/3 = 9
- 7/1 + 8/4 = 9

EXPLORATION: 6 has no solutions.

Puzzle of the Week

Fractions – 7

THE CHALLENGE: Use the numbers 1 to 9 at most once each to fill in these boxes. Do not use improper fractions. How many solutions can you find?

$$\frac{\square}{\square} + \frac{\square}{\square} = \frac{\square}{\square}$$

1 2 3 4 5 6 7 8 9

Puzzle of the Week

Fractions – 7 – Notes

THE CHALLENGE: Playing around with compatible denominators produces this list:

- 1/6 + 4/8 = 2/3
- 1/6 + 3/9 = 2/4
- 1/4 + 2/8 = 3/6
- 1/6 + 3/9 = 4/8
- 1/2 + 3/9 = 5/6
- 1/3 + 2/4 = 5/6
- 1/3 + 4/8 = 5/6
- 2/4 + 3/9 = 5/6
- 4/8 + 3/9 = 5/6

Puzzle of the Week

Fractions – 8

THE CHALLENGE: Use the numbers 1 to 9 at most once each to fill in these boxes. Organize your answers by putting the fractions in increasing size of their denominators. Can you find more than one solution?

$$\frac{\square}{\square} + \frac{\square}{\square} + \frac{\square}{\square} = 1$$

1 2 3 4 5 6 7 8 9

EXPLORATION: Can you solve it if 1 is replaced by other numbers, such as 2 or 3? What happens if you insist that the number on the right side of the equation cannot be one of the numbers on the left side?

Puzzle of the Week

Fractions – 8 – Notes

THE CHALLENGE & EXPLORATION: Here are the solutions for 1.

- 1/6 + 4/8 + 3/9 = 1
- 1/4 + 3/6 + 2/8 = 1
- 2/4 + 1/6 + 3/9 = 1

Here are the 9 solutions for 2.

- 1/2 + 3/4 + 6/8 = 2
- 1/2 + 7/6 + 3/9 = 2
- 1/3 + 2/4 + 7/6 = 2
- 1/3 + 7/6 + 4/8 = 2
- 2/3 + 5/6 + 4/8 = 2
- 1/4 + 9/6 + 2/8 = 2
- 5/4 + 3/6 + 2/8 = 2
- 2/4 + 7/6 + 3/9 = 2
- 7/6 + 4/8 + 3/9 = 2

Here are the 13 solutions for 3.

- 2/1 + 3/6 + 4/8 = 3
- 2/1 + 4/6 + 3/9 = 3
- 1/2 + 4/3 + 7/6 = 3
- 1/2 + 6/3 + 4/8 = 3
- 4/2 + 1/3 + 6/9 = 3
- 1/2 + 8/4 + 3/6 = 3
- 1/2 + 7/4 + 6/8 = 3
- 5/2 + 1/6 + 3/9 = 3
- 7/3 + 2/4 + 1/6 = 3
- 1/3 + 8/4 + 6/9 = 3
- 7/3 + 1/6 + 4/8 = 3
- 5/4 + 9/6 + 2/8 = 3
- 9/4 + 3/6 + 2/8 = 3

If the number on the right cannot be one of the numbers on the left, these are the only solutions from the earlier ones that still work.

- 1/3 + 7/6 + 4/8 = 2
- 7/6 + 4/8 + 3/9 = 2
- 1/2 + 7/4 + 6/8 = 3
- 5/4 + 9/6 + 2/8 = 3

Puzzle of the Week
Fractions – 9

THE CHALLENGE: Use the numbers 2 to 9 at most once each to fill in these boxes. There are a lot of answers that are essentially the same, so organize them with increasing numerators and denominators.

$$\frac{\square}{\square} \times \frac{\square}{\square} = 1$$

2 3 4 5 6 7 8 9

Puzzle of the Week

Fractions – 9 – Notes

THE CHALLENGE & EXPLORATION: Write the equation with variables to make it easier to talk about: A/B × C/D = 1. Multiplying both sides by B and D turns this into A × C = B × D.

As there is only one 5 and one 7, they cannot be put into this equation. Thinking in terms of primes helps a lot. We need to balance the number of 2's and 3's on both sides of this equation using the remaining six numbers: 2, 3, 4, 6, 8, and 9.

The 3's are more limited, so there are only three possibilities.

- There are no multiples of 3. This only leaves three numbers, which is not enough.
- 3 is on one side and 6 is on the other side. We will need to use two of the three remaining numbers.
  - 3 x 4 = 6 x 2; 3 x 8 = 6 x 4
- 9 is on one side and 3 and 6 are on the other side.
  - 9 x 2 = 3 x 6

So, there are three solutions (plus three more if you flip the fractions):

- (3 x 4) / (2 x 6)
- (3 x 8) / (4 x 6)
- (2 x 9) / (3 x 6)

Puzzle of the Week
Fractions – 10

THE CHALLENGE: Use the numbers 2 to 9 at most once each to fill in these boxes. There are a lot of answers that are essentially the same, so organize them with increasing numerators and denominators.

$$\frac{\square}{\square} \times \frac{\square}{\square} = \square$$

2 3 4 5 6 7 8 9

Puzzle of the Week

Fractions – 10 – Notes

THE CHALLENGE & EXPLORATION: Write the equation with variables to make it easier to talk about: A/B × C/D = E. Multiplying both sides by B and D turns this into A × C = B × D × E.

As there is only one 5 and one 7, they cannot be put into this equation. Thinking in terms of primes helps a lot. We need to balance the number of 2's and 3's on both sides of this equation using the remaining six numbers: 2, 3, 4, 6, 8, and 9.

The 3's are more limited, so there are only three possibilities.

- There are no multiples of 3. This only leaves three numbers, which is not enough.
- 3 is on one side and 6 is on the other side. We will need to use all three of the remaining numbers. Unfortunately, none of these cases leads to a solution.
- 9 is on one side and 3 and 6 are on the other side.
    - 9 on the left: 9 x 4 = 3 x 6 x 2; 9 x 8 = 3 x 6 x 4
    - 9 on the right: does not work

So, there are two possible ways for A x C = B x D x E. In each of these ways, if you isolate one of the numbers on the right side and make it E, you can divide by the other two numbers on the right side and get a solution to the original problem. You can pair up the answers by putting the numerators and denominators in increasing order..

- 2: (4 x 9) / (3 x 6)
- 3: (4 x 9) / (3 x 6); (8 x 9) / (4 x 6)
- 4: (8 x 9) / (3 x 6)
- 6: (4 x 9) / (2 x 3); (8 x 9) / (3 x 4)

Puzzle of the Week
Fractions – 11

THE CHALLENGE: Use the numbers 1 to 9 at most once each in these boxes. Put the numerators and denominators in increasing order to keep your solutions orderly.

$$\frac{\square}{\square} \times \frac{\square}{\square} \times \frac{\square}{\square} = 1$$

1 2 3 4 5 6 7 8 9

Puzzle of the Week

Fractions – 11 – Notes

THE CHALLENGE: Write the equation with variables to make it easier to talk about: A/B × C/D × E/F = 1. Similar to the Fractions – 9 puzzle notes, multiply by the denominators to make this A x C x E = B x D x F.

As there is only one 5 and one 7, they cannot be put into this equation. Thinking in terms of primes helps a lot. We need to balance the number of 2's and 3's on both sides of this equation using the remaining seven numbers: 1, 2, 3, 4, 6, 8, and 9 to fill the six slots.

The 3's are more limited, so there are only two possibilities when solving A x C x E = B x D x F.

- 3 is on the left side and 6 is on the right side. We need to use all four of the remaining numbers. This is impossible to do. Including 6, there are seven factors of 2 in all the numbers - one 2 for 2 and 6, two 2's for the 4, and three 2's for the 8. So, it is impossible to split these seven things (2's) evenly between the two sides.
- 9 is on the left side and 3 and 6 are on the right side. We will be leaving out one of the remaining four numbers to complete the equation. We have seven 2's to deal with, and leaving out one number must give an even number of 2's - we can either leave out the 2 or the 8.
  - 9 x 1 x 8 = 3 x 6 x 4
  - 9 x 1 x 4 = 3 x 6 x 2

So, up to switching denominators around, we have four solutions (where the second two are just the flip of the first two):

- (1 x 8 x 9) / (3 x 4 x 6)
- (1 x 4 x 9) / (2 x 3 x 6)
- (3 x 4 x 6) / (1 x 8 x 9)
- (2 x 3 x 6) / (1 x 4 x 9)

Puzzle of the Week

Fractions – 12

THE CHALLENGE: Use the numbers 1 to 9 at most once each in each set of boxes. First, make $\frac{\square}{\square} \times \frac{\square}{\square}$ equal to 2/3, and then find values that make $\frac{\square}{\square} \times \frac{\square}{\square}$ as close as possible to 5/11.

$$\frac{\square}{\square} \times \frac{\square}{\square} = \frac{2}{3}$$

$$\frac{\square}{\square} \times \frac{\square}{\square} \sim \frac{5}{11}$$

1 2 3 4 5 6 7 8 9

Puzzle of the Week

Fractions – 12 – Notes

THE CHALLENGE: I will write the expression as A/B x C/D to make it easier to talk about.

Part 1: A/B x C/D = 2/3. Multiply both sides by 3 x B x D, so this becomes 3 x A x C = 2 x B x D. Look at this using the prime factors 2 and 3 – these must balance on the two sides of the equation. From 3, 6, 9, we have four factors of 3 to work with. From 2, 4, 6, 8 we have seven factors of 2 to work with. There are two cases for handling the 3's.

- Case 1: Neither A nor C is 3 or 6, and B is 3 or 6.
  - B=3: 3 x A x C = 2 x 3 x D reduces to A x C = 2 x D. Solutions to this are 1 x 4 = 2 x 2, 1 x 8 = 2 x 4.
  - B=6: 3 x A x C = 2 x 6 x D reduces to A x C = 4 x D. The solution to this is 1 x 8 = 4 x 2.
- Case 2: A is 3 or 6, and B is 9.
  - A=3: 3 x 3 x C = 2 x 9 x D reduces to C = 2 x D - the options for (C, D) are (2, 1), (4, 2), and (8, 4).
  - A=6: 3 x 6 x C = 2 x 9 x D reduces to C = D, which is impossible.

So, the solutions to the problem are given by:

- (1 x 4) / (3 x 2)
- (1 x 8) / (3 x 4)
- (1 x 8) / (6 x 2)
- (3 x 2) / (9 x 1)
- (3 x 4) / (9 x 2)
- (3 x 8) / (9 x 4)

Part 2: A/B x C/D ~ 5/11. Multiply both sides by 11 x B x D to turn this into 11 x A x C ~ 5 x B x D. We want to find multiples of 11 that are close to multiples of 5 and that can be produced using 1 to 9.

Here is the list to consider. These are multiples that are 1 apart and are producible using 1 to 9: 11 x 4 ~ 8 x 5; 11 x 9 ~ 20 x 5; and 11 x 16 ~ 35 x 5. The bigger the numbers are, the better, so let's look at this last one.

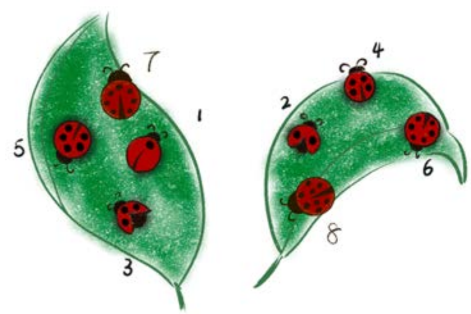
(2 x 8) / (5 x 7) = 16 / 35 is very close to 5/11 - they differ by 1 / (35 x 11) = 1 / 385.

Perhaps the multiples that are two or three apart will work out better? There do not seem to be any candidates that would be an improvement.

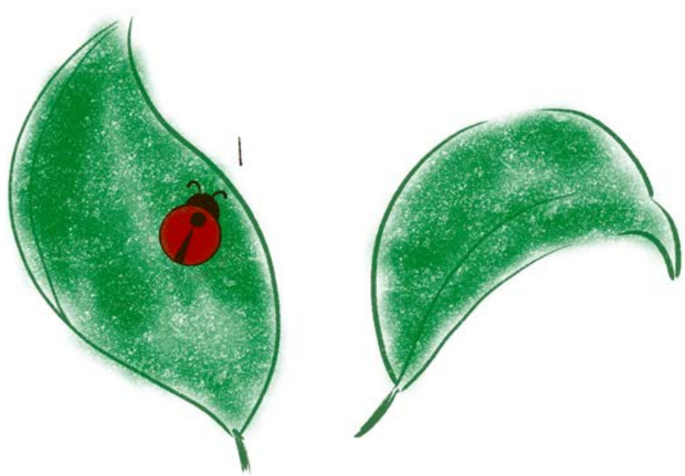
Puzzle of the Week

Ladybugs that don't Multiply

Numbered ladybugs are landing on two leaves. The rule is that no two ladybugs on a leaf can multiply to be the number of another ladybug on that leaf. The leaf on the left is fine, but the leaf on the right has 2 x 4 = 8.



THE CHALLENGE: Starting at 1 and counting up, how high can you go putting the numbered ladybugs on either of the two leaves while following the rule for each of the leaves.



EXPLORATION: What are large sets of (not necessarily consecutive) numbers that are allowable to have on a single leaf? How much higher can you go with consecutive numbers if you use more than two leaves?

Puzzle of the Week

Ladybugs that don't Multiply – Notes

THE CHALLENGE & EXPLORATION: This puzzle provides an opportunity to look more closely at primes and prime factorizations.

One leaf can have all numbers that are units (1), primes, and squares of primes. That's a very big chunk of numbers. The other leaf gets all the rest of the numbers. Doing that gets all the numbers up to 47, and it's hard to see how to improve on that. Here are the numbers for the two leaves:

(1, 2, 3, 4, 5, 7, 9, 11, 13, 17, 19, 23, 25, 29, 31, 37, 41, 43, 47} and {6, 8, 10, 12, 14, 15, 16, 18, 20, 21, 22, 24, 26, 27, 28, 30, 32, 33, 34, 35, 36, 38, 39, 40, 42, 44, 45, 46}

What the best strategy to use for three leaves is far from clear. One simple approach is to take numbers that are products on the second leaf and put them on a third leaf - numbers such as 48, 60, 72, 80, and 84. The second leaf would hold all things of the form $p^3, p^4, p^5, p^6, p\ x\ q, p^2 x\ q,\ p^3 x\ q,\ p^2 x\ q^2, p\ x\ q\ x\ r$, where p, q, and r are primes - that's a large and safe collection. This process would get you all the way up to the number 48 x 60 - 1, so that's a very large range to work with.

Another approach would be to break apart the second leaf's numbers into two groups, but it's not clear how best to do that.

Puzzle of the Week

Letter Substitutions – 11

Rules:

1. A letter represents a digit from 0 to 9, and has the same value throughout a single puzzle.
2. No number can start with the digit 0.
3. Within a puzzle, different letters must have different values.

$$\begin{array}{r} 8 \\ +\ A \\ \hline B\ 2 \end{array} \Rightarrow \begin{array}{r} 8 \\ +\ 4 \\ \hline 1\ 2 \end{array}$$

THE CHALLENGE: Find the value of A, B, C, D, and E to make this puzzle work.

$$\begin{array}{r} A\ B\ C\ D\ E \\ \times \qquad\qquad A \\ \hline E\ E\ E\ E\ E\ E \end{array}$$

EXPLORATION: Make some letter substitution puzzles for others to solve.

Puzzle of the Week

Letter Substitutions – 11 – Notes

THE CHALLENGE: The key to solving this is to focus on A x E = E, 1E, 2E, or whatever is needed before the E. There are very few combinations of single digits that have this property. Note that A cannot be 0 or 1, and E cannot be 0. Here is the list of the possibilities, listing as A, E:

3, 5: 3 x 5 = 15

5, 1: 5 x 1 = 5

5, 3: 5 x 3 = 15

5, 7: 5 x 7 = 35

5, 9: 5 x 9 = 45

7, 5: 7 x 5 = 35

9, 5: 9 x 5 = 45

Once you have this list, it is quick work to see that A = 7 and E = 5 is the only one that works. This gives the solution A = 7, B = 9, C = 3, D = 6, and E = 5. The finished puzzle looks like this:

```
  7 9 3 6 5
x         7
-----------
5 5 5 5 5 5
```

Puzzle of the Week

Letter Substitutions – 12

Rules:

1. A letter represents a digit from 0 to 9, and has the same value throughout a single puzzle.
2. No number can start with the digit 0.
3. Within a puzzle, different letters must have different values.

$$\begin{array}{r} 8 \\ +\ \ A \\ \hline B\ \ 2 \end{array} \quad \Rightarrow \quad \begin{array}{r} 8 \\ +\ \ 4 \\ \hline 1\ \ 2 \end{array}$$

THE CHALLENGE: Find the value of S, A, T, U, R, N, P, L, and E to make this puzzle work.

$$\begin{array}{ccccccc} & S & A & T & U & R & N \\ + & U & R & A & N & U & S \\ \hline P & L & A & N & E & T & S \end{array}$$

EXPLORATION: Make some letter substitution puzzles for others to solve.

Puzzle of the Week

Letter Substitutions – 12 – Notes

THE CHALLENGE: Because carrying in adding problems with two numbers can be at most 1, we know that P must be 1. Also, N + S = S forces N = 0.

With those values, the puzzle becomes:

```
  S A T U R 0
+ U R A 0 U S
1 L A 0 E T S
```

Looking at U + 0 + (possible carry) = E tells us that E is one more than U, there is a carry from the previous column, and there is no carry to the next column. That then means that T + A = 10, and there is a carry to the next column. A + R + (carry of 1) = 1A means that R is 9 and there is a carry to the next column.

We now know E = U + 1, T + A = 10, and the puzzle looks like this:

```
  S A T U 9 0
+ U 9 A 0 U S
1 L A 0 E T S
```

Looking at the tens column, 9 + U + (no carry) = 1T means that T is one less than U. Consequently, we have three digits in a row: T, U, and E.

Look through the possibilities using T + A = 10. The remaining question is how to make S + U + (carry) = 1L?

- T = 2, U = 3, E = 4, A = 8. Unused so far: 5, 6, 7. It’s not possible to solve S + 3 + 1 = 1L.
- T = 3, U = 4, E = 5, A = 7. Unused so far: 2, 6, 8. It’s not possible to solve S + 3 + 1 = 1L.
- T = 4, U = 5, E = 6, A = 6. Impossible with A = E.
- T = 5, U = 6, E = 7, A = 5. Impossible with A = T.
- T = 6, U = 7, E = 8, A = 4. Unused so far: 2, 3, 5. S = 5 and L = 3 works!

We end up with S = 5, A = 4, T = 6, U = 7, R = 9, N = 0, P = 1, L = 3, and E = 8. The solution looks like this:

```
  5 4 6 7 9 0
+ 7 9 4 0 7 5
1 3 4 0 8 6 5
```

Puzzle of the Week

Letter Substitutions – 13

Rules:

1. A letter represents a digit from 0 to 9, and has the same value throughout a single puzzle.
2. No number can start with the digit 0.
3. Within a puzzle, different letters must have different values.

```
    8          8
+   A   =>  +  4
 ----       ----
B   2       1  2
```

THE CHALLENGE: Find the value of P, O, T, A, M, L, E, U, and C to make this puzzle work.

```
    P  O  T  A  T  O
+   T  O  M  A  T  O
 -------------------
 L  E  T  T  U  C  E
```

EXPLORATION: Make some letter substitution puzzles for others to solve.

Puzzle of the Week

Letter Substitutions – 13 – Notes

THE CHALLENGE: Because carrying in adding problems with two numbers can be at most 1, we know that L must be 1.

Note that in the ones column O + O = E and that in the ten thousands column O + O + (possible carry) = T. So the "possible carry" is a definite carry. Next notice one column to the right that we have T + M + (possible carry) = 1T. If M is 0, the sum will be less than 10. Therefore, M = 9 and the possible carry is a definite carry. Note also that O + O = E and O + O + 1 = T forces T = E + 1.

With those values, the puzzle becomes:

```
  P O T A T O
+ T O 9 A T O
1 E T T U C E
```

P + T + (possible carry) = E and T = E + 1 forces P to be 8 or 9. However, 9 is taken. So P = 8 and, once again, the possible carry is a definite carry. Given the carry information we have established, we also know that O + O is at least 10 and A + A is at least 10. Put another way, O and A are at least 5.

To summarize, here is our updated puzzle, and we also know T = E + 1, C = T + T + 1, and O and A are at least 5.

```
  8 O T A T O
+ T O 9 A T O
1 E T T U C E
```

At this point, we have 0, 2, 3, 4, 5, 6, and 7 to work with, and we need to find the value of E, T, C, O, A and U. Note that O + O = E forces E to be even.

- E = 2, T = 3, C = 7, O = 6.
- E = 4, T = 5, C = 1, O = 7. This is impossible as it forces C = L = 1.
- E = 6, T = 7, C = 5, O = 8. This is impossible as it forces O = P = 8.

Therefore E = 2, T = 3, C = 7, O = 6, P = 8, M = 9, and L = 1. That only leaves 0, 4, and 5 for A and U. Fortunately, A = 5 and U = 0 works, and we are done! Here is the finished puzzle:

```
  8 6 3 5 3 6
+ 3 6 9 5 3 6
1 2 3 3 0 7 2
```

Puzzle of the Week

Letter Substitutions – 14

THE CHALLENGE: Using each of the numbers from 1 to 9 exactly once, find the value for each of these letters that will make both of these equations true.

$$AB \times C = DE$$

$$F \times G = KL$$

EXPLORATION: Is there more than one solution?

Puzzle of the Week

Letter Substitutions – 14 – Notes

THE CHALLENGE & EXPLORATION: Start by narrowing down which letters might be 5.

Because 5 times anything will end in 5 or 0, neither of which is possible, 5 cannot be in the ones digits of any of the numbers. That only leaves A, D, and K as places for 5.

If A = 5, then AB is at least 51. C must be at least 2 (or AB would equal DE). In this case AB times C would be larger than a two-digit number. So, A cannot be 5.

For the remaining cases, we need to know the factorizations of all the numbers in the 50's, so here they are:

- 51 = 3 x 17 (repeats 1)
- 52 = 2 x 26 (repeats 2) = 4 x 13
- 53 = prime
- 54 = 2 x 27 (repeats 2) = 3 x 18 = 6 x 9
- 55 = 5 x 11 (repeats 1)
- 56 = 2 x 28 (repeats 2) = 4 x 14 (repeats 4) = 7 x 8
- 57 = 3 x 29
- 58 = 2 x 29 (repeats 2)
- 59 = prime

If D = 5, there are only three possibilities that don't repeat digits:

- 52 = 4 x 13 - This leaves 6, 7, 8, 9 for FGKL, which cannot work.
- 54 = 3 x 18 - This leaves 2, 6, 7, 9 for FGKL, which cannot work.
- 57 = 3 x 29 - This leaves 1, 4, 6, 8 for FGKL, which cannot work.

If K = 5, there are only two possibilities that don't repeat digits:

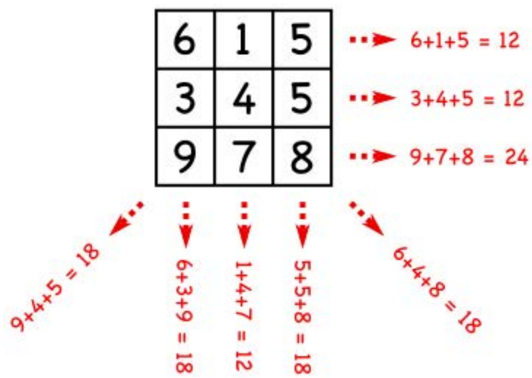
- 54 = 6 x 9 - This leaves 1, 2, 3, 7, 8 for ABCDE, and 27 x 3 = 81 !!!!
- 56 = 7 x 8 - This leaves 1, 2, 3, 4, 9 for ABCDE, which cannot work.

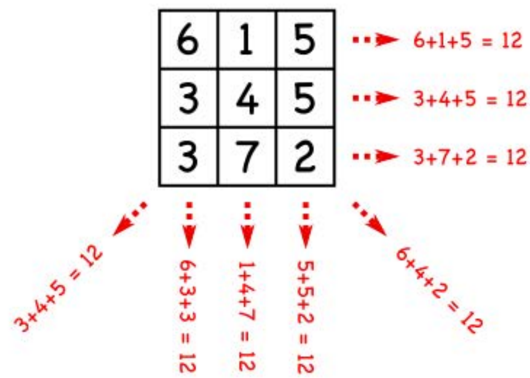
There is only one solution (ignoring swapping F and G) and it is:

27 x 3 = 81 and 6 x 9 = 54

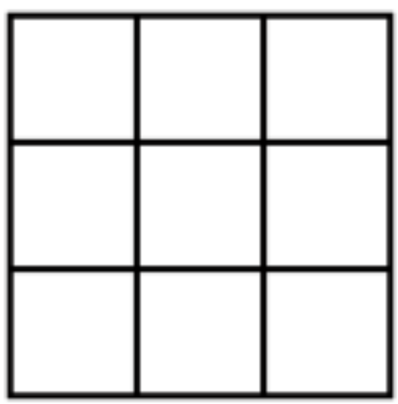
Puzzle of the Week
Magic Squares – 7

In a traditional ***Magic Square***, all the rows, columns and diagonals add up to the same number. This first square is not a Magic Square. The second one is a Magic Square with a constant sum of 12.





THE CHALLENGE: Make a Magic Square, without duplicate entries, that uses multiplication instead of adding. That is, this will be a Magic Square where the entries of each of the rows, columns, and diagonals ***multiply*** to give the same product.



EXPLORATION: If you can make one type of puzzle look like another type of puzzle you already know how to solve, you can save a lot of work and get some interesting insights. Were you able to use your knowledge of Adding Magic Squares to help you make Multiplying Magic Squares?

Puzzle of the Week

Magic Squares – 7 – Notes

THE CHALLENGE & EXPLORATION: With two key ideas, these puzzles become very easy.

The first idea is to use prime factorizations. If you prime factor all the numbers, you will quickly see that what happens for one prime has no effect on what happens for all the other primes. Take this solution as an example.

$2^8 3$	2^1	$2^6 3^2$
$2^3 3^2$	$2^5 3$	2^7
2^4	$2^9 3^2$	$2^2 3$

Thinking multiplicatively, this can be split into two squares that get multiplied by each other term by term. Notice that what happens in the powers of 2 Magic Square does not influence and is not dependent at all on what happens in the powers of 3 Magic Square.

2^8	2^1	2^6
2^3	2^5	2^7
2^4	2^9	2^2

3	1	3^2
3^2	3	1
1	3^2	3

The second idea becomes evident looking at these squares. The exponents in each square form an additive Magic Square!

To summarize, to create a multiplicative Magic Square, make multiplicative Magic Squares for each prime you want to involve, and then multiply together those squares. To make the squares for the individual primes, use additive Magic Squares for the exponents.

Here is a final example that uses smaller numbers. The common product is 216.

12	1	18
9	6	4
2	36	3

Puzzle of the Week
Moving Digits – 4

THE CHALLENGE: Find a six-digit number whose product by 2, 3, 4, 5, and 6 results in numbers that use the same six digits.

EXPLORATION: If you have worked with decimal values for fractions, do the six products from The Challenge remind you of any fractions?

Puzzle of the Week

Moving Digits – 4 – Notes

THE CHALLENGE: Represent x, the six-digit number as ABCDEF.

A = 1: Because 6 x ABCDEF is a six-digit number, A must be 1. Any other value would make 6x bigger than a six-digit number.

A, B, C, D, E, F are all different: Multiply ABCDEF by 1 through 6 with results between 100000 and 999999 forces the high-order digits of 1x, 2x, 3x, 4x, 5x, and 6x to start at 1 and end no higher than 9, and therefore they must all be different.

None of A, B, C, D, E, and F is 0: Each one of these digits takes a turn at being the high-order digit, so they can never be 0.

F is odd and not 5: If F were even, then 5x would end in 0. If F = 5, then 2x would end in 0.

Each of A, B, C, D, E, and F takes a turn at being a ones digit: A consequence of F being odd and not equal to 5 is that the ones digits of the products 1x, 2x, 3x, 4x, 5x, and 6x are all different. Therefore, the ones digits must take on the six values for A through F.

F = 7: 1 is one of the values, so one of the multiples has 1 as a ones digits. The only multiple where that can happen is 3x. Because 3x has a ones digit of 1, F must be 7.

The ones digits in order are 7, 4, 1, 8, 5, and 2: This comes from looking at the products using F = 7.

The numbers have the form 1BCDE7, 2xxxx4, 4xxxx1, 5xxxx8, 7xxxx5, and 8xxxx2. This is essentially a summary of what we have so far.

B = 4: If B = 2, then 6x would be less than 800000. If B = 5, then 6x would be more than 899999.

The remaining digits are 2, 5, and 8. Some trial and error with filling in the remaining places ends up with the result.

The answer is: 142857.

EXPLORATION: This digit sequence is the same as the decimal representation for 1/7. If you look at the decimal representations for 2/7, 3/7, up to 6/7, each representation contains the same sequence of six numbers only starting at a different place in the sequence. Pretty amazing!

Puzzle of the Week

Pan Balance With Weights – 1

A pan balance tells you when its two sides are carrying the same amount of weight or whether one side is heavier than the other.

THE CHALLENGE: You have a very large collection of 4-ounce and 7-ounce weights to use on one side of a pan balance. By using two 4-ounce weights, you can measure an 8-ounce item. Which weights can you weigh exactly and which ones can't you weigh exactly?

EXPLORATION: How do your results change if you have 5- and 9-ounce weights? How about other pairs of weights that have no common divisor larger than 1? Can you find any patterns in your data?

Puzzle of the Week

Pan Balance With Weights – 1 – Notes

THE CHALLENGE: This is often called the Chicken McNugget Theorem.

Start by doing something not entirely obvious. Make a chart of the numbers with rows of length one of the two numbers you're working with. We'll make the rows 7 long, but it would work just as well to make them 4 long. Next, put all the sums of multiples of 4 and multiples of 7 in red, as shown below.

1	2	3	**4**	5	6	**7**
8	9	10	**11**	**12**	13	**14**
15	**16**	17	**18**	**19**	**20**	**21**
22	**23**	**24**	**25**	**26**	**27**	**28**
29	**30**	**31**	**32**	**33**	**34**	**35**

A few things jump out when you see the data this way. One is that, once a column has one hit, the rest of that column is filled. Another is that the multiples of 4 bounce around the columns without repetition until you hit 4 x 7. By 4 x 7, every column has been hit by a multiple of 4.

Starting at 18, all the numbers are hit. This is in line with the general theorem. The theorem says that if the two numbers are m and n and they are relatively prime, then all numbers will be hit starting with (m - 1) x (n - 1), which in our case is 6 x 3 = 18.

Another part of the theorem is that exactly half the numbers from 1 to (m - 1) x (n - 1) will be hit. In our case that is 9 numbers out of 18.

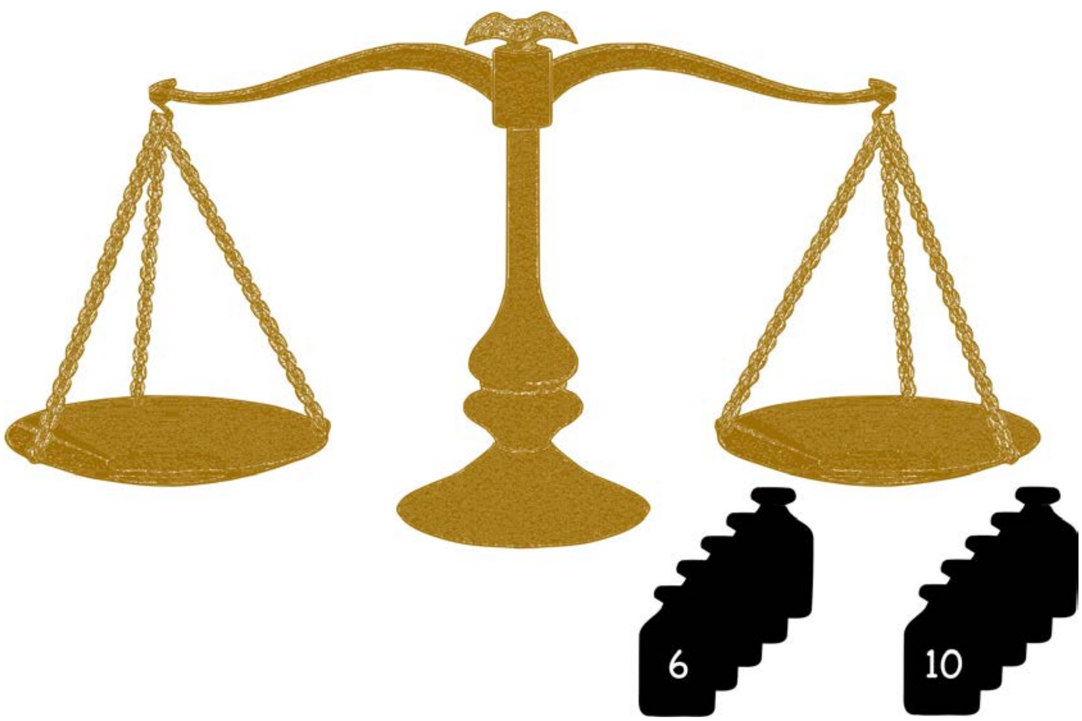
EXPLORATION: For 5 and 9, the point of saturation starts at (5 - 1) x (9 - 1) = 4 x 8 = 32, and 16 of the numbers up to 32 will be hit.

Puzzle of the Week

Pan Balance With Weights – 2

A pan balance tells you when its two sides are carrying the same amount of weight or whether one side is heavier than the other.

THE CHALLENGE: You have a very large collection of 6-ounce and 10-ounce weights to use on one side of a pan balance. By using two 6-ounce weights, you can measure a 12-ounce item. Which weights can you weigh exactly and which ones can't you weigh exactly?



EXPLORATION: How do your results change if you have 6- and 9-ounce weights? How about other pairs of weights that have a common divisor larger than 1? How do your results compare to the ones you got in "Pan Balance With Weights - 1"? Can you make use of that earlier work to save you reinventing things for this problem?

Puzzle of the Week

Pan Balance With Weights – 2 – Notes

THE CHALLENGE: The only difference between this puzzle and "Pan Balance With Weights - 1" is that the two numbers have a common divisor bigger than 1. In mathematics, we seek to take advantage of earlier work whenever we can.

For 6 and 10, the greatest common divisor is 2. Every multiple of either number will have a factor of 2, and so will all of their sums. One consequence of this is that any number that is not a multiple of 2 can never be weighed by our weights.

To take advantage of our earlier work, create a new weight - call it the TwoOunce. Now our weights are 3 TwoOunces and 5 TwoOunces. The advantage of doing that is that our numbers now have a greatest common divisor of 1, and we can use all of our earlier work. We can weigh all of the TwoOunce weights starting at (3 -1) x (5 - 1) TwoOunces, and we can weigh half of the TwoOunce weights up to that point.

Translating that result into ounces gives: We can weigh all the weights that are multiple of two ounces starting at 2 x 4 x 2 = 16 ounces, and half of the two-ounce multiples up to 16 ounces will be measurable.

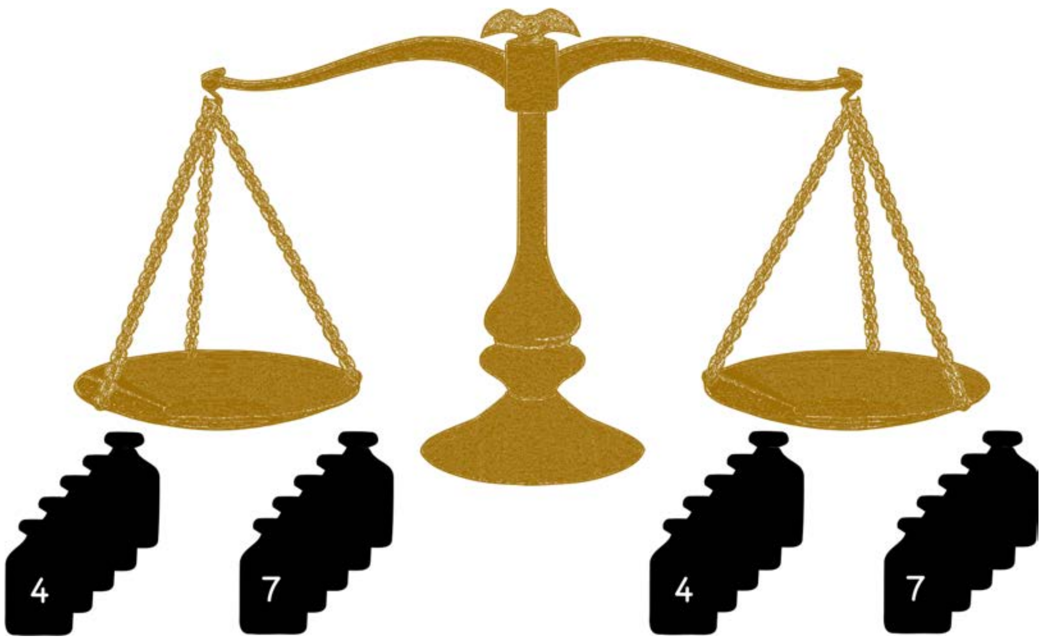
EXPLORATION: For 6 and 9, the greatest common multiple is 3. So, only multiples of 3 ounces can possibly be hit, and all multiples of 3 ounces will be hit starting with 3 x (2 - 1) x (3 - 1) = 3 x 1 x 2 = 6 ounces.

Puzzle of the Week

Pan Balance With Weights – 3

A pan balance tells you when its two sides are carrying the same amount of weight or whether one side is heavier than the other.

THE CHALLENGE: You have a very large collection of 4-ounce and 7-ounce weights to use on both sides of a pan balance. You can weigh a 3-ounce item by using a 4-ounce weight together with the item on one side and a 7-ounce weight on the other. Which weights can you weigh exactly and which ones can't you weigh exactly?



EXPLORATION: How do your results change if you have 5- and 9-ounce weights? How about other pairs of weights that have no common divisor larger than 1? Can you find any patterns in your data? What would happen if you had three kinds of weights to work with - say 3 ounces, 4 ounces, and 7 ounces?

Puzzle of the Week

Pan Balance With Weights – 3 – Notes

THE CHALLENGE: Allowing weights on both sides of the pan balance is like allowing positive and negative multiples of both numbers. For example, suppose we weigh an item by putting the item with one 7-ounce weights on one side and three 4-ounce weights on the other side. The equation is <item> + 1 x 7 = 3 x 4. Putting the multiple of 7 on the other side gives <item> = 3 x 4 - 1 x 7 = 5. We have created a negative multiple of 7 by putting it on the same pan with the item.

Therefore, we are looking at all numbers that can be created by adding any multiple (positive, zero, or negative) of 4 to any multiple of 7. This is called Bezout's Theorem. The theorem states that the set of all possible sums of the multiples of two numbers is exactly the set of every multiple of their greatest common divisor. Because the greatest common divisor of 4 and 7 is 1, in this case we can weigh every multiple of 1 ounce, which is all numbers.

One way to see that this is true is to use the Chicken McNugget Theorem from "Pan Balance with Weights - 1." Suppose the two numbers are n and m. From that theorem, we know that using only nonnegative multiples, we can hit every weight starting with (n - 1) x (m - 1). In particular, we can write the weight (n x m) - 1 as a sum of a multiple of n and a multiple of n. Then 1 = [(n x m) - 1] - n x m tells us that 1 can be written as a sum of multiples of n and m.

EXPLORATION: Because 5 and 9 have a greatest common divisor of 1, we can weigh all amounts with these weights too.

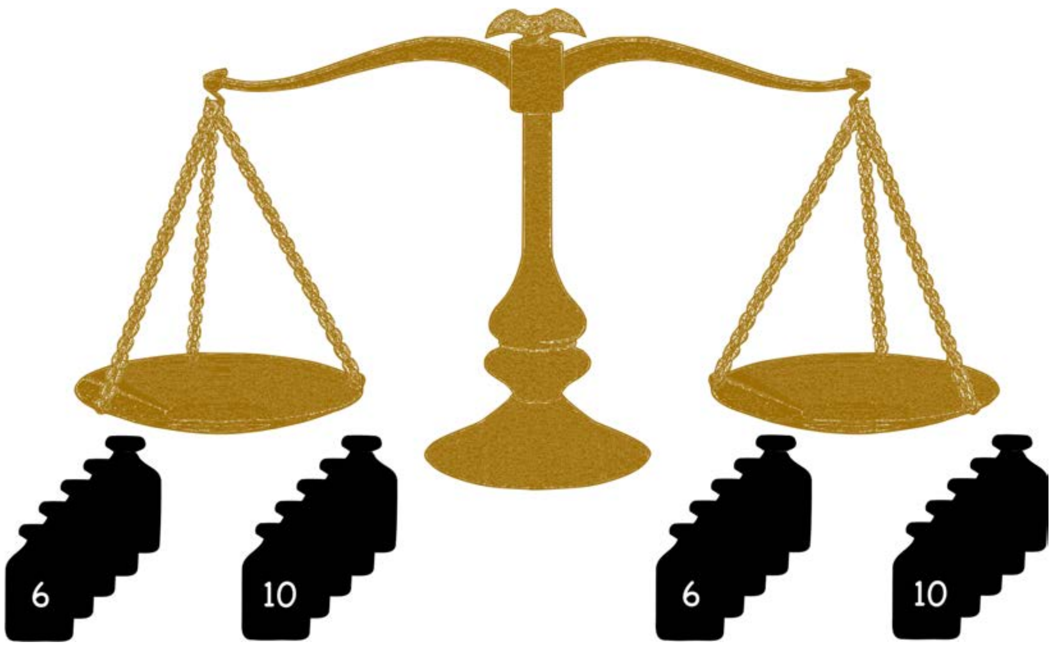
If you had three weights that had at least one pair of numbers that had a greatest common divisor of 1, you would be able to weigh all possible weights and you would have more choices of how to do it.

Puzzle of the Week

Pan Balance With Weights – 4

A pan balance tells you when its two sides are carrying the same amount of weight or whether one side is heavier than the other.

THE CHALLENGE: You have a very large collection of 6-ounce and 10-ounce weights to use on both sides of a pan balance. You can weigh a 4-ounce item by using a 6-ounce weight together with the item on one side and a 10-ounce weight on the other. Which weights can you weigh exactly and which ones can't you weigh exactly?



EXPLORATION: How do your results change if you have 6- and 9-ounce weights? How about other pairs of weights that have a common divisor larger than 1? How do your results compare to the ones you got in "Pan Balance With Weights - 3"? Can you make use of that earlier work to save you reinventing things for this problem? What would happen if you had three kinds of weights to work with - say 3 ounces, 6 ounces, and 10 ounces?

Puzzle of the Week

Pan Balance With Weights – 4 – Notes

THE CHALLENGE: We can employ the same idea we used in going from Pan Balance With Weights 1 to 2. Because the greatest common divisor of 6 and 10 is 2, everything will be a multiple of 2. So, Bezout's Theorem (mentioned in Pan Balance With Weights - 3) tells us that we will be able to weigh all objects that have a weight that is a multiple of 2.

EXPLORATION: Because the greatest common divisor of 6 and 9 is 3, we will be able to weigh all multiples of 3 ounces with those two weights.

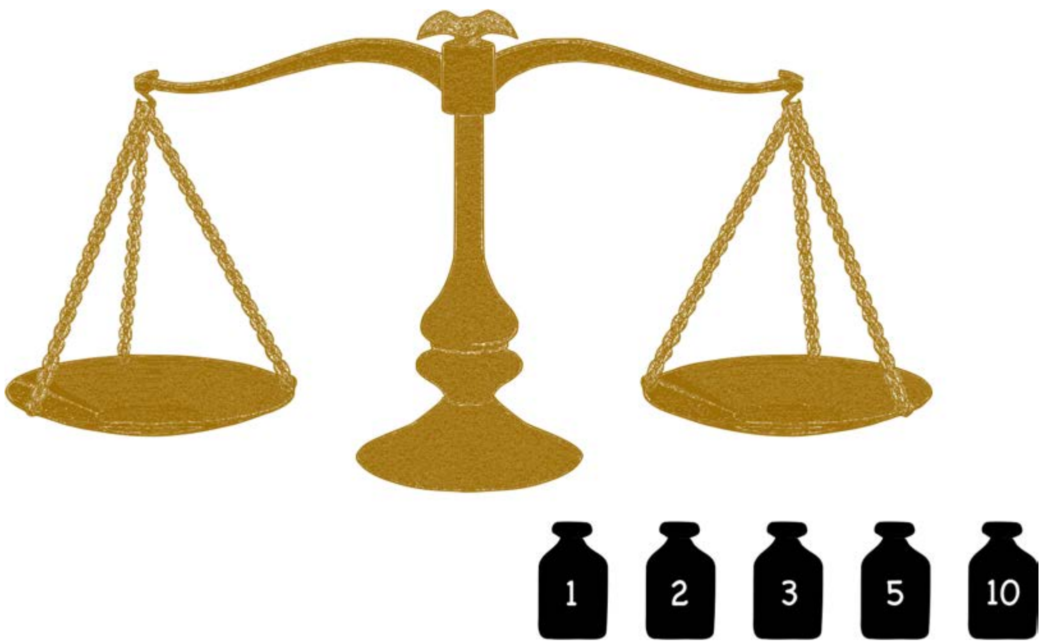
If there are three weights, start by analyzing what happens with a pair of them. We saw that the multiples of 6 and 10 give us all multiples of 2. Then we can then take all multiples of 2 and 3 (by combining 6 and 10) and see that we are able to get all multiples of 1 (all numbers).

Puzzle of the Week

Pan Balance With Weights – 5

A pan balance tells you when its two sides are carrying the same amount of weight or whether one side is heavier than the other.

THE CHALLENGE: You would like to be able to weigh anything that is a whole number of ounces from 1 ounce up to 15 ounces. If you can only put the weights on one side of the pan balance, what is the fewest number of weights that will make this work and what will the weights be?



EXPLORATION: How does your answer change if you want to be able to weigh anything between 1 ounce and 25 ounces? How about other ranges? Are there some ranges where there is only one best answer, and other ranges where there are many?

Puzzle of the Week

Pan Balance With Weights – 5 – Notes

THE CHALLENGE: The simplest answer is to always use powers of 2. This uses the ideas behind the base 2 number system, though you do not need to tell your students about that. To be able to weigh all amounts from 1 to 15, use the four weights 1, 2, 4, and 8.

EXPLORATION: For the numbers that are a power of 2 minus 1, there is only one answer - the powers of 2.

For other numbers, there is a choice. For example, if you want to cover the numbers from 1 to 14, you can use 1, 2, 4, and 8, but you can also use 1, 2, 4, and 7.

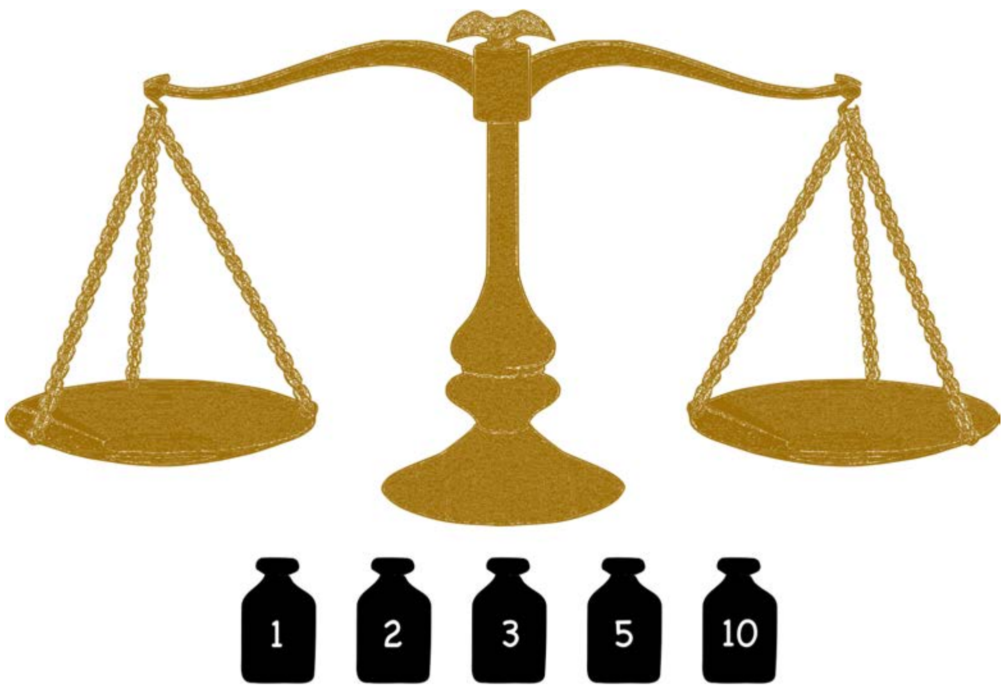
A nice thing about using powers of 2 is that there is always exactly one way to weigh each amount. If you use 1, 2, 4, and 7, then a weight of 7 can be weighed using one weight of 7 or using the three weights 1, 2, and 4.

Puzzle of the Week

Pan Balance With Weights – 6

A pan balance tells you when its two sides are carrying the same amount of weight or whether one side is heavier than the other.

THE CHALLENGE: You would like to be able to weigh anything that is a whole number of ounces from 1 ounce up to 40 ounces. If you can put weights on either side or both sides of the pan balance, what is the fewest number of weights that will make this work and what will the weights be?



EXPLORATION: How does your answer change if you want to be able to weigh anything between 1 ounce and 50 ounces? How about other ranges? Are there some ranges where there is only one best answer, and other ranges where there are many?

Puzzle of the Week

Pan Balance With Weights – 6 – Notes

THE CHALLENGE: The ideas behind this puzzle take advantage of the base 3 number system, though you do not need to tell your students about that.

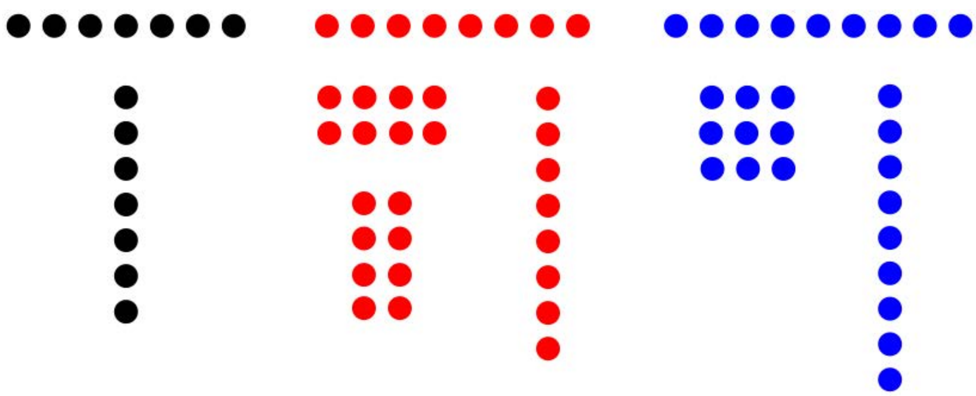
The key to this problem is to use the powers of 3. The four numbers 1, 3, 9, and 27, when placed on one side or the other of the scale, can weigh every amount from 1 to 40. For example, if the item weighs 7 ounces, it can be weighed by doing <item> + 3 on one side and 1 + 9 on the other.

EXPLORATION: As with "Pan Balance With Weights - 5," if the range is not at the maximal amount, then there is flexibility, especially with the largest number in the set. The maximal amount is determined by adding up the powers of three. For example, three weights would have a maximum of 1 + 3 + 9 = 13, and four weights would be 1 + 3 + 9 + 27 = 40, which is the problem we started with.

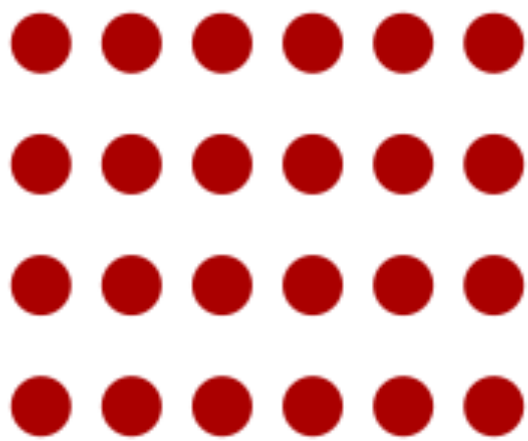
Puzzle of the Week

Rectangle Shapes

Here are all the rectangles that can be made with 7, 8, and 9 dots.



THE CHALLENGE: What are all the rectangles you can make with 23, 24, 25, 26, and 27 dots? How many different shapes of rectangles can you make with each of these sizes?



EXPLORATION: Which types of numbers have exactly two rectangles, both of which are flat? Which numbers have exactly three rectangles?

Puzzle of the Week

Rectangle Shapes – Notes

THE CHALLENGE: Each rectangle represents one way to factor the number. The number of rectangles will be the number of divisors of the number.

Here are the factorings (= rectangles) for each number.

- 23: 1 x 23, 23 x 1
- 24: 1 x 24, 2 x 12, 3 x 8, 4 x 6, 6 x 4, 8 x 3, 12 x 2, 24 x 1
- 25: 1 x 25, 5 x 5, 25 x 1
- 26: 1 x 26, 2 x 13, 13 x 2, 26 x 1
- 27: 1 x 27, 3 x 9, 9 x 3, 27 x 1

EXPLORATION: The numbers with exactly two factorings are the prime, such as 7 and 23. They can be factored as having just one row or just one column.

The numbers with exactly three factorings are squares of primes, such as 9 and 25. In addition to the two flat rectangles, they have one more that is a square that has the prime number of dots on each side.

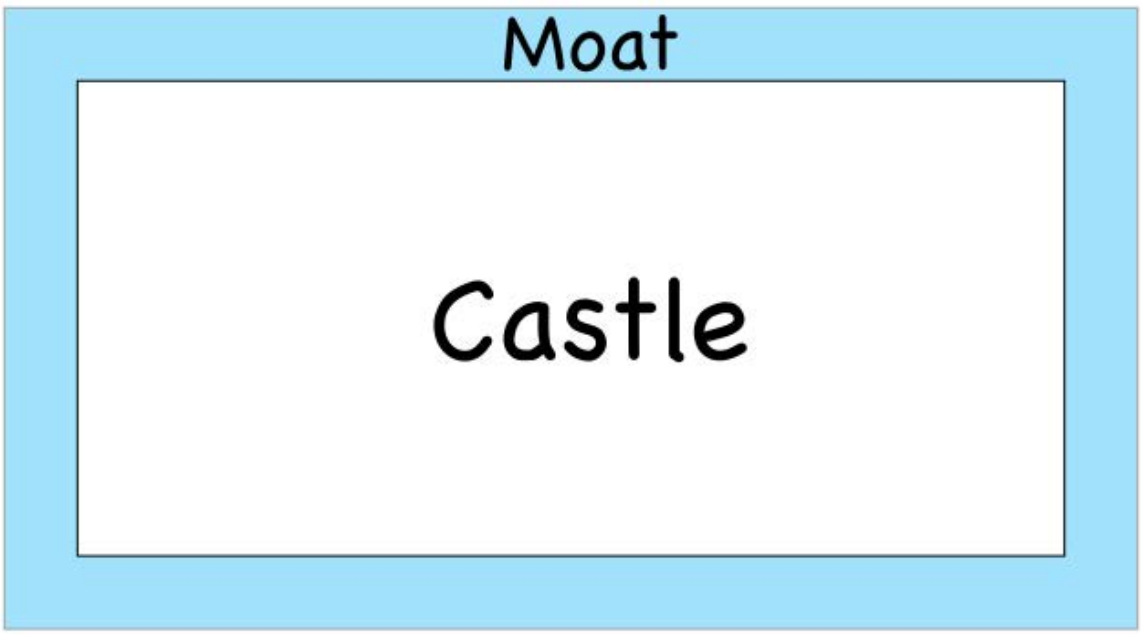
There is a lot more that can be done to count divisors. For example, it is very easy to count the divisors of prime powers. For example, 5 to the 10th power will have 11 divisors - one for each power of 5 from 0 to 10.

There is a lot more exploration of divisors that is possible for the interested student.

Puzzle of the Week

Crossing the Moat

THE CHALLENGE: You are trying to break into a rectangular castle that has a very deep and dangerous moat with rigid sides that is 8 meters wide all around it. Through a miscalculation, you have two very strong boards that are each 20 centimeters wide by 775 centimeters long. You have no nails or any other way to attach or tie the boards to each other. How can you use the boards to walk across the moat?



The moat is 8 meters wide.

The boards are 20cm by 775cm

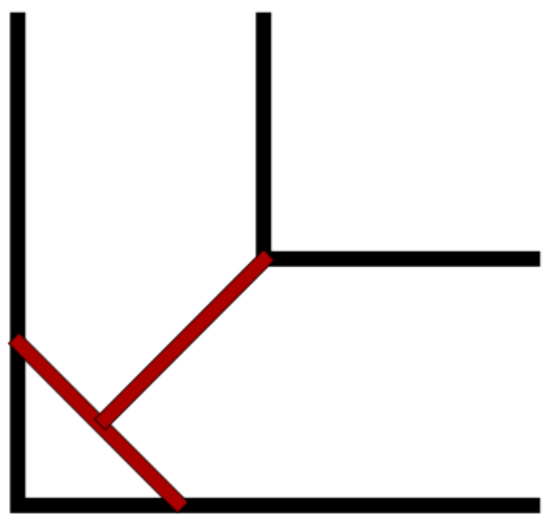
Puzzle of the Week

Crossing the Moat – Notes

THE CHALLENGE: They should place the boards in a corner of the moat as shown below.

If your students know some geometry and the Pythagorean Theorem, they can calculate the total diagonal is $8\sqrt{2} \cong 11.3$ meters. The middle of the first board will be about 7.75 / 2 = 3.875 meters of the way along the diagonal. So when 7.75 meters is added to that we get 3.875 + 7.75 = 11.625 meters, which is enough for a little overlap for stability.

If your students don't know the necessary geometry, they can still make a scale drawing with a moat that is 8 centimeters wide and then make measurements with a ruler to see that the boards would work in the corner. They can even cut out pieces of paper the right scaled size to simulate the boards.



Puzzle of the Week

Squares Made of Primes

THE CHALLENGE: For numbers that are squares bigger than 1, some squares are the sum of two prime numbers. For example, 4 = 2 + 2 and 9 = 2 + 7. Is there a square bigger than 1 that is not the sum of two prime numbers? If so, what is the smallest square, bigger than 1, that is not a sum of two primes?

$$n^2 = p + q?$$

EXPLORATION: What are some observations that can make your search more efficient? What happens if you are allowed to use more than two primes?

Puzzle of the Week

Squares Made of Primes – Notes

THE CHALLENGE: Goldbach's Conjecture says that all even numbers larger than 2 are the sum of two primes. This conjecture is unproven, but no one has found a counterexample, and people have searched to very large numbers (all numbers up to $4\ x\ 10^{18}$). So, all even squares are probably the sum of two primes, and we should check the odd squares.

To have two primes add up to an odd number, one of the primes must be 2. This makes checking very easy.

9 = 2 + 7

25 = 2 + 23

49 = 2 + 47

81 = 2 + 79

121 = 2 + 119 and 119 = 7 x 17

EXPLORATION: It has been proven that every even number starting with 4 is the sum of at most four primes. By adding "3" to that result, we know that every number is the sum of at most five primes. Someone thinks they have proven that every odd number is the sum of three primes.

So, it seems very likely that every even square is the sum of two primes and every odd square is the sum of either two or three primes, depending on whether $n^2 - 2$ is a prime. For example, 121 = 3 + 5 + 113.